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# Interfacial tension of the chiral Potts model

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**Abstract.** We obtain the interfacial tension of the general solvable $N$-state chiral Potts model. It has exponent $\mu = (N+2)/(2N)$ in both the horizontal and vertical directions, in agreement with the scaling relation $2\mu = 2 - \alpha$.

## 1. Introduction

The 'chiral Potts model' is a planar lattice model with $N$-state spins that live on the sites of the lattice and interact along edges. The 'solvable' case is when the interactions are chosen as in [1] so that the star-triangle or 'Yang–Baxter' relations are satisfied. If the lattice is oriented diagonally, as in figure 1, then this ensures that the row-to-row transfer matrices commute. Relying on our experience of previously solved models, we expect that for this case it should be possible to calculate the bulk free energy and some other large-lattice properties, such as the correlation length and interfacial tension. The free energy was calculated in [2], and more explicitly in [3, 4].

The model is 'superintegrable' when the vertical rapidities satisfy a particular relation, causing the transfer matrix eigenvalues to simplify. The associated Hamiltonian is Hermitian. This case has been studied extensively [5–18], particularly when $N = 3$, and the correlation lengths and interfacial tensions have been obtained for particular boundary conditions [10]. Until now they have not been calculated for the more general solvable ferromagnetic model with real positive Boltzmann weights, which we refer to as the 'physical' or real model: here we calculate the vertical interfacial tension $\epsilon_r$, using the results of a previous paper [19]. The horizontal tension follows at once by rotation.

We use a 'Z-invariance' argument [20] that $\epsilon_r$ should be independent of the vertical rapidities. We therefore only need to calculate it for the superintegrable case. Unfortunately the previous results for this case [10] are not applicable, being either in the wrong direction or with inappropriate boundary conditions. Here we calculate the appropriate $\epsilon_r$, imposing skewed boundary conditions so as to force a vertical interface, and using previous results [19]. It has the correct 180° rotation symmetry, and in the scaling region is unchanged by 90° rotations (in fact it is then independent of all rapidities, being a function only of the temperature variable $k'$). The critical exponent is

$$\mu = \frac{1}{N} + \frac{1}{2}. \tag{1}$$

A curious feature is that we expect our results to be true for the general physical model all the way to the order–disorder transition, even though the superintegrable case (which has

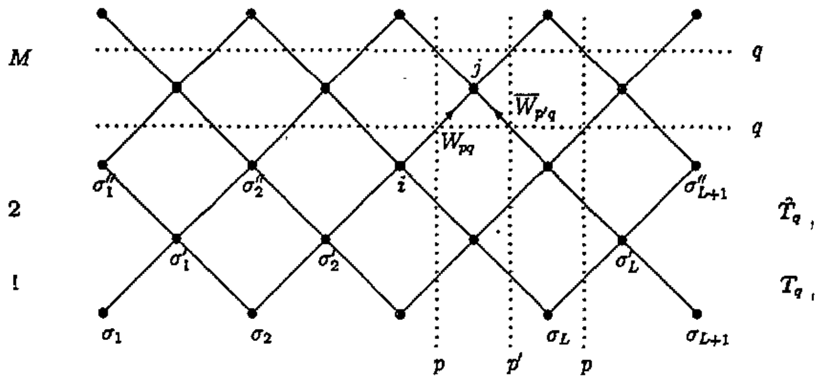


**Figure 1.** The square lattice $\mathcal{L}$ of $2M$ rows with $L$ sites per row. $T_q$ is the transfer matrix of an odd row, $\hat{T}_q$ of an even row.

complex Boltzmann weights) may have an earlier transition to an incommensurate phase [7, 9, 21]. This is because we expect our arguments to be valid for sufficiently small $k'$ and there to be no non-analyticities in the physical model as $k'$ approaches the critical value $k' = 1$.

Our results for the scaling region show that it is energetically unfavourable to interpose an intermediate phase between two others, i.e. that there is no wetting. This agrees with the low-temperature results of [19]. Presumably there is no wetting at any sub-critical temperature. This appears to conflict with the suggestion in [22, 23] that the solvable chiral Potts is the wetting transition of the Ostlund–Huse model. As we mention at the end of this paper, this may be explained by the interfacial tension being different in the diagonal direction from the horizontal and vertical.

## 2. The model

Consider the square lattice $\mathcal{L}$, drawn diagonally as in figure 1, with $L$ sites per row. At each site $i$ there is a spin $\sigma_i$, which takes values $0, \ldots, N-1$. We impose periodic boundary conditions at the top and bottom of $\mathcal{L}$; and skewed conditions at the right and left boundaries, so that $\sigma_{L+1}, \sigma_1$ in the figure are related by $\sigma_{L+1} = \sigma_1 - r$, and similarly for $\sigma''_{L+1}, \sigma''_1$, etc. Here $r$ is some fixed integer such that $0 \leqslant r < N$.

Let $k$ be a real constant, $0 < k < 1$, $k' = (1-k^2)^{1/2}$, and let $\omega = \exp(2\pi i/N)$, $\omega^{j/2} = \exp(\pi i j/N)$ for any integer $j$. Let a 'rapidity' $q$ be a set of complex numbers $\{a_q, b_q, c_q, d_q\}$, related by

$$a_q^N + k' b_q^N = k d_q^N \qquad k' a_q^N + b_q^N = k c_q^N . \tag{2}$$

For any two rapidities $p$, $q$, and any integer $n$, define functions (periodic in $n$ of period $N$)

$$W_{pq}(n) = \prod_{j=1}^{n} \frac{d_p b_q - \omega^j a_p c_q}{b_p d_q - \omega^j c_p a_q} \qquad \overline{W}_{pq}(n) = \prod_{j=1}^{n} \frac{\omega a_p d_q - \omega^j d_p a_q}{c_p b_q - \omega^j b_p c_q} . \tag{3}$$

With each horizontal dotted line in figure 1 we associate a rapidity $q$, with the vertical lines rapidities $\ldots, p, p', p, p', \ldots$ alternately as indicated. For edges of $\mathcal{L}$ intersected by a vertical $p$-line, adjacent spins interact with Boltzmann weight $W_{pq}(\sigma_i - \sigma_j)$ on SW $\rightarrow$ NE edges, and $\overline{W}_{pq}(\sigma_i - \sigma_n)$ on SE $\rightarrow$ NW edges, as indicated. For the other edges, $p$ is replaced by $p'$.

The partition function is defined in the usual way as

$$Z_r = \sum \prod (\text{edge weights}) \tag{4}$$

the sum being over all values of all the spins, and the product over the weights of all the edges of $\mathcal{L}$. If $\mathcal{L}$ has $2M$ rows (and hence $2LM$ sites) this can be written [24] as

$$Z_r = \text{Trace } (\mathbf{T}_q \widehat{\mathbf{T}}_q)^M \tag{5}$$

where $\mathbf{T}_q$ and $\widehat{\mathbf{T}}_q$ are the transfer matrices of the two types of row, as in [19, 25]. Because of the star-triangle relation [1], $\mathbf{T}_q$ and $\widehat{\mathbf{T}}_q$ commute, in the sense that $\mathbf{T}_q \widehat{\mathbf{T}}_s \propto \mathbf{T}_s \widehat{\mathbf{T}}_q$ and $\widehat{\mathbf{T}}_q \mathbf{T}_s \propto \widehat{\mathbf{T}}_s \mathbf{T}_q$, for all rapidities $q$, $s$.

Let $T_q$ and $\widehat{T}_q$ be eigenvalues of $\mathbf{T}_q$ and $\widehat{\mathbf{T}}_q$, i.e. $\mathbf{T}_q \boldsymbol{y} = T_q \boldsymbol{x}$, $\widehat{\mathbf{T}}_q \boldsymbol{x} = \widehat{T}_q \boldsymbol{y}$, where $\boldsymbol{x}, \boldsymbol{y}$ are vectors (independent of $q$). Then

$$Z_r = \sum (T_q \widehat{T}_q)^M \tag{6}$$

the sum being over all eigenvalues. The interfacial tension $\epsilon_r$ between phases $\sigma$ and $\sigma - r$ can be defined by [19]

$$\epsilon_r / K\mathcal{T} = - \lim_{L, M \to \infty} M^{-1} \ln(Z_r / Z_0) . \tag{7}$$

Here $K$ is Boltzmann's constant and $\mathcal{T}$ the temperature, and the limit has to be taken so that $L$ and $M$ both tend to infinity together. We expect the system to be ferromagnetically ordered, the degree of order decreasing to zero as $k'$ increases from 0 to 1, with an order–disorder transition at $k' = 1$. Thus $k'$ is a temperature-like parameter.

Other rapidity variables that we shall use are

$$x_q = a_q / d_q \qquad y_q = b_q / c_q \qquad \mu_q = d_q / c_q \qquad t_q = x_q y_q \qquad \Lambda_q = \mu_q^N . \tag{8}$$

They are related by

$$k x_q^N = 1 - k' / \Lambda_q \qquad k y_q^N = 1 - k' \Lambda_q \qquad [(\Lambda_q - 1)/(\Lambda_q + 1)]^2 = (\eta^N - t_q^N)/(\eta^{-N} - t_q^N) \tag{9}$$

where

$$\eta = [(1 - k')/(1 + k')]^{1/N} . \tag{10}$$

Two further variables are $u_q$, $v_q$, related to one another and the others by

$$\cos N v_q = k \cos N u_q \qquad t_q = \mathrm{e}^{2\mathrm{i} u_q} \qquad x_q = \mathrm{e}^{\mathrm{i}(u_q - v_q)} \qquad y_q = \mathrm{e}^{\mathrm{i}(u_q + v_q)} \tag{11}$$
$$\Lambda_q = -\mathrm{i}\, \mathrm{e}^{\mathrm{i}N v_q} \sin N(u_q + v_q) / (k' \cos N u_q) = \mathrm{i}\, \mathrm{e}^{\mathrm{i}N v_q} k' \cos N u_q \,/ \sin N(u_q - v_q) .$$

If $u_q$ is real, then we can choose $v_q$ to be real, between 0 and $\pi/N$. It follows that $\mathrm{i}\Lambda_q \mathrm{e}^{-\mathrm{i}N v_q}$ is positive real, so we can choose $\mu_q \exp[\mathrm{i}(2v_q - \pi/N)/2]$ also to be positive real. With these choices, $x_q, y_q, \mu_q$ are single-valued analytic functions of $u_q$ for $u_q$ real. They can be continued analytically as single-valued functions of $u_q$ throughout the horizontal strip $\mathcal{D}$:

$$|\mathrm{Im}(u_q)| < \tfrac{1}{2} \ln \eta \tag{12}$$

in the complex $u_q$-plane. In this strip $0 < \mathrm{Re}(v_q) < \pi/N$.

The point $u_q = -\frac{1}{2}\mathrm{i} \ln \eta$ (which is when $t_q = \eta$) is a branch point of $v_q$. We shall need to locally extend $\mathcal{D}$ around this point, so that in this vicinity it becomes a two-sheeted Riemann surface. The value of $v_q$ on one surface is minus the value on the other. At the branch point $v_q$ is zero. From now on we regard $u_p, u_{p'}, u_q$ as lying in this extended domain $\mathcal{D}$.

### 2.1. Real case

If $u_p, u_{p'}, u_q$ are all real, $v_p, v_{p'}, v_q$ all lie in the interval $(0, \pi/N)$, and

$$u_q - \pi/N < u_p, u_{p'} < u_q \tag{13}$$

then the Boltzmann weights have the physical property that they are all real and positive, and we expect the partition function, free energy and interfacial tensions to be analytic functions of $u_p, u_{p'}, u_q$.

### 2.2. Hermitian case

Another interesting case is when $u_p, u_{p'}, u_q - \pi/(2N)$ are all pure imaginary (the imaginary parts of $u_p, u_{p'}$ being numerically less than $\frac{1}{2}\ln\eta$), $v_p, v_{p'}, \mu_q$ are real, $v_q - \pi/(2N)$ is pure imaginary, and $|\mu_p| = |\mu_{p'}| = 1$. Then $x_q, \omega^{-1/2}y_q, \omega^{-1/2}t_q, \Lambda_q$ are real and positive, $\Lambda_q > 1/k'$, $y_p = x_p^*$ and $\overline{W}_{pq}(-n) = W_{pq}(n)^*$ (similarly with $p \to p'$). It follows that $\widehat{\mathbf{T}}_q$ is the Hermitian conjugate of $\mathbf{T}_q$, so the two-row transfer matrix $\mathbf{T}_q\widehat{\mathbf{T}}_q$ is Hermitian and its eigenvalues $T_q\widehat{T}_q$ are real. This 'Hermitian case' intersects with the real one at the 'symmetric point' $u_p = u_{p'} = 0$, $u_q = v_q = \pi/(2N)$.

### 2.3. Superintegrable case

The 'alternating superintegrable' case is when the vertical rapidities $p$ and $p'$ relations

$$u_{p'} = u_p \qquad v_{p'} = -v_p\,. \tag{14}$$

Then $x_{p'} = y_p, y_{p'} = x_p, \Lambda_{p'} = 1/\Lambda_p$ and the functional form of the eigenvalues $T_q$, $\widehat{T}_q$ then simplifies greatly, facilitating the solution of the functional relations. This case has been discussed (particularly the homogeneous sub-case, when $u_{p'} = u_p = -\frac{1}{2}\mathrm{i}\eta$ and $v_{p'} = v_p = 0$) in a sequence of papers [5–18], and in [19]. It can be realized only in the extended domain $\mathcal{D}$ defined above, and has no intersection with the real case. It does, however, have an intersection with the Hermitian case. It can be illuminating to focus on this Hermitian sub-case: many of the formulae of section 3, in particular (37) and (36), are then explicitly real.

### 2.4. Z-invariance and analyticity

For the real case, using Z-invariance arguments [20], we expect the interfacial tensions to be independent of $p$ and $p'$. They are analytic, so this should also be true for the Hermitian case, at least provided $u_p, u_{p'}, u_q$ are sufficiently close to the real axis. Further, from low-temperature calculations [19] and consideration of the $N = 2$ case (when the chiral Potts model reduces to the well known Ising model), at least for sufficiently small $k'$ we expect them to be analytic if we move $u_p$ and $u_{p'}$ off the real axis to the neighbourhood of the branch point $u_p = -\frac{1}{2}\mathrm{i}\ln\eta$, or even round this branch point on to the other Riemann sheet of the function $v_p$ (so long as we do not then move too far from the branch point—we do expect non-analyticities when $u_p$ is real on the other sheet).

It follows that the interfacial tensions should be independent of $p$ and $p'$ along such paths, so we need only consider the 'alternating superintegrable' case (14) taking $u_p = u_{p'}$ to be close to $-\frac{1}{2}\mathrm{i}\ln\eta$, while $u_q$ remains on or near the real axis, between 0 and $\pi/N$, with $|\Lambda_q| > 1$. *The result obtained should be true, not only for the superintegrable case, but also for the real one.* It should also hold for the Hermitian case, provided the imaginary parts of $u_p, u_{p'}, u_q$ are not too large.

There is a difficulty in applying the results to the superintegrable case itself near the critical point $k' = 1$, since McCoy and his co-workers have argued strongly that there can then be transitions to incommensurate phases, causing completely different behaviour. Nevertheless, since we do not expect any phase transitions or non-analyticities for the physical case as $k'$ increases from 0 to 1, we expect our results to be valid for the physical case all the way to the ferromagnetic order–disorder transition at $k' = 1$.

## 3. Eigenvalues $T_q, \hat{T}_q$ for the superintegrable case

In [8, equation (2.22)] and [18, equation (2.21)] McCoy *et al* have proposed an ansatz for the form of $T_q$ and $\widehat{T}_q$ for the homogeneous superintegrable case (when $x_p = x_{p'} = y_p = y_{p'}$). This has been verified, and generalized to the case (14) with skewed boundary conditions, in [19, section 6]. Let $F(x)$ be a polynomial of degree $m_P$,

$$F(x) = \prod_{j=1}^{m_P} (1 + \omega v_j x) \tag{15}$$

where the constants $v_1, \ldots, v_{m_P}$ are given by

$$\left(\frac{v_j + \omega^{-1}}{v_j + \omega^{-2}}\right)^L = \omega^{-P_a - P_b} \prod_{l=1}^{m_P} \frac{v_j - \omega^{-1} v_l}{v_j - \omega v_l} \qquad j = 1, \ldots, m_P \tag{16}$$

$P_a$, $P_b$ being some integers.

Define a related function

$$\mathcal{P}(z) = \omega^{-P_b} \sum_{j=0}^{N-1} \frac{(1 - z^N)^L \, (\omega^j z)^{-P_a - P_b}}{(1 - \omega^j z)^L \, F(\omega^j z) \, F(\omega^{j+1} z)} . \tag{17}$$

Plainly this is a rational function of $z$. One can verify from (16) that it has no finite non-zero poles, so it is a Laurent polynomial. It is invariant under $z \to \omega z$, so it is a Laurent polynomial in $z^N$. $P_a$ and $P_b$ are to be chosen so that $\mathcal{P}(0)$ is finite and non-zero, so $\mathcal{P}(z)$ is strictly a polynomial in $z^N$. Let its zeros be $-\lambda_1, \ldots, -\lambda_{m_E}$ (i.e. $\mathcal{P}(z) = 0$ when $z^N = -\lambda_j$). Define

$$w_k = [(\eta^{-N} + \lambda_k t_p^N)/(\eta^N + \lambda_k t_p^N)]^{1/2} \tag{18}$$

$$D = t_p^{NL-L-P_a-P_b} \, \mathcal{P}(\eta / t_p) \tag{19}$$

$$G(\Lambda) = D^{1/2} \prod_{k=1}^{m_E} \{[\Lambda + 1 \pm (\Lambda - 1) w_k]/(2\Lambda)\} . \tag{20}$$

Then $G(0), G(k'), G(1/k')$ are non-zero and

$$T_q = N^{L/2} \frac{(x_q - y_p)^L}{(x_q^N - y_p^N)^L} \, {x_q}^{P_a} \, {y_q}^{P_b} \, \mu_q^{-P_a} \, F(t_q/t_p) \, G(\Lambda_q) \tag{21}$$

and $\widehat{T}_q$ is given similarly, but with $y_p$ replaced by $x_p$. (We have replaced the $G(\Lambda)$ of [19] by $G(1/\Lambda)$, and re-scaled the functions $F(x), \mathcal{P}(x)$.) The Boltzmann weights remain finite when any one of $a_q, b_q, c_q, d_q$ tend to zero. Hence so does $T_q$ and it follows that the integers $P_a$, $P_b$, $P_\mu$ are non-negative, satisfying

$$P_b + m_P \leqslant P_\mu \leqslant (N-1)L - P_a - Nm_E - m_P . \tag{22}$$

They must also satisfy the selection rules

$$P_\mu = r \mod N \tag{23}$$

$$P_b - P_a = Q + r + L \mod N \tag{24}$$

where $\omega^Q$ is the eigenvalue of the spin-shift operator $X$ of [19, 25].

The $m_E$ sign choices in (20) are independent. We have obtained the normalization factor $D$ by taking $t_q = \eta$, $\Lambda_q = 1$ in [19, equation (6.24)].

For the Hermitian sub-case, the eigenvalues of $T_q\widehat{T}_q$ are all real, which from (21) implies that the $w_k$ are real, for all positive $t_p$. This can only happen if the $\lambda_k$ are all real and positive. Since the $\lambda_k$ are independent of $p$ and $q$, this must be true for the superintegrable model in general.

### 3.1. Maximum eigenvalues

To evaluate $Z_r$ for $M$ large, we need only the largest eigenvalues $T_q$, $\widehat{T}_q$. We focus on the case mentioned above, when $u_p = u_{p'}$ is close to $-\frac{1}{2}\mathrm{i}\ln\eta$ and $u_q$ is close to the real axis, between 0 and $\pi/N$. Then the RHS of (20) is maximized by choosing all the signs to be plus.

The dominant eigenvalues have been calculated in [19] for the low-temperature limit when $k' \to 0$, $x_p, y_p, x_{p'}, y_{p'}, x_q \to 1$, $y_q \to t_q$. Then $\Lambda_q \to \infty$ and $G(\Lambda_q) \to 1$. From (5.19) therein, we see that these are the eigenvalues (21), with

$$P_b = 0 \qquad P_\mu = m_P = r . \tag{25}$$

Further, using the variables $\alpha_j$, $\chi$ defined in [19], and the relation $v_j = -\omega^{-1}\mathrm{e}^{-2\mathrm{i}\alpha_j}$ given before (6.23) therein, $v_1, \ldots, v_r$ form a single string:

$$v_j = s\,\omega^{j-2-r/2} \qquad j = 1, \ldots, r \tag{26}$$

where $s = \mathrm{e}^{2\mathrm{i}\chi}$. Thus $F(x) = A(\omega^{-1/2}sx)$, where $A(x)$ is the polynomial

$$A(x) = \prod_{j=1}^{r}(1 + \omega^{j-(r+1)/2}x) . \tag{27}$$

The relations (16) are satisfied, provided only that

$$\left(\frac{\omega^{r/2}s + 1}{s + \omega^{r/2}}\right)^L = \omega^{-Qr-rL/2} . \tag{28}$$

Since $\chi$ is pure imaginary, $s$ is real and positive.

A related polynomial that we shall need is

$$h(x) = A(\omega^{-1/2}x)\,A(\omega^{1/2}x) = \prod_{j=-r/2}^{r/2}(1 + \omega^j x)^{n_j} \tag{29}$$

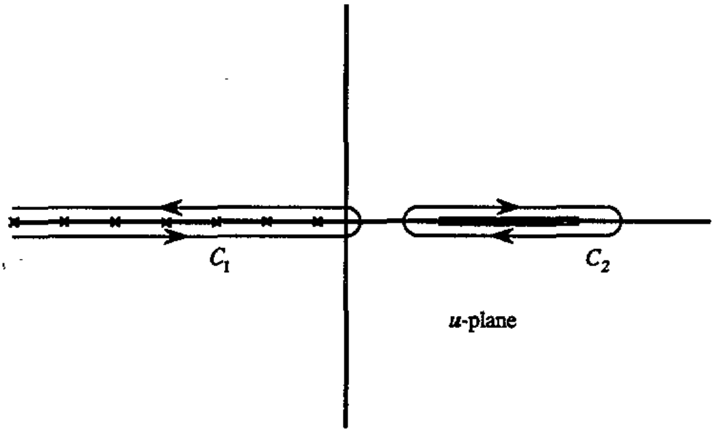


**Figure 2.** The equivalent contours $C_1$, $C_2$ in the $u$ plane. The heavy line denotes the branch cut from $\eta^N t_p^{-N}$ to $\eta^{-N} t_p^{-N}$.

where (for $r > 0$) $n_j = 1$ when $j = \pm r/2$, else $n_j = 2$. $A(x)$ and $h(x)$ are real when $x$ is real.

Also define

$$g(\Lambda, x) = \ln\left[\frac{\Lambda+1}{2\Lambda} + \frac{\Lambda-1}{2\Lambda}\left(\frac{\eta^{-N}+x^N}{\eta^N+x^N}\right)^{1/2}\right]. \tag{30}$$

Using Cauchy's residue theorem, equation (20) can be written

$$\ln G(\Lambda) = \frac{1}{2}\ln D + \frac{1}{2\pi \mathrm{i}}\int_{C_1} g(\Lambda, \omega^{1/2} z t_p)\left(\frac{\mathrm{d}}{\mathrm{d}u}\ln \mathcal{P}(z)\right)\mathrm{d}u \tag{31}$$

where $u = z^N$ and $C_1$ surrounds the negative real axis in the complex $u$-plane, on which the zeros $u = -\lambda_k$ of $\mathcal{P}(z)$ are located, as in figure 2.

Because $g(\Lambda, \infty) = 0$, the integrand in (31) tends to zero faster than $1/u$ as $u \to \infty$, so $C_1$ can be closed round a full 360° circle at $\infty$ so as to enclose the whole $u$-plane apart from the negative real axis. Provided $t_p$ has an argument between $-\pi/N$ and $\pi/N$, and $|\Lambda| > 1$, the only singularity of the integrand now within $C_1$ is a branch cut (arising from the square root in from (30)) from $\eta^N/t_p^N$ to $\eta^{-N}/t_p^N$. (The argument of the logarithm in (30) has no zeros in this cut plane.) Hence $C_1$ can now be shrunk down to the contour $C_2$ in figure 2, just surrounding the branch cut. Changing the variable of integration to $z$ and defining

$$\psi(\Lambda, x) = \tan^{-1}\left[\frac{\Lambda-1}{\Lambda+1}\left(\frac{\eta^{-N}-x^N}{x^N-\eta^N}\right)^{1/2}\right] \qquad \eta < x < \eta^{-1} \tag{32}$$

we obtain

$$\ln G(\Lambda) = \tfrac{1}{2}\ln D + \frac{1}{\pi}\int_S \frac{\mathcal{P}'(z)}{\mathcal{P}(z)}\,\psi(\Lambda, z t_p)\,\mathrm{d}z \tag{33}$$

the integration now being along the straight-line segment $S = (\eta/t_p, \eta^{-1}/t_p)$.

### 3.2. The limit $L \to \infty$

The expression (33) is exact, for finite $L$. When $L$ becomes large and $z$ lies on $S$, the sum in (17) is dominated by its $j = 0$ term, the other terms being exponentially smaller in $L$. Similarly, $D$ is given by (19) and (17), the sum in (17) again being dominated by the $j = 0$ term. Hence

$$D = \left(\frac{t_p^N - \eta^N}{t_p - \eta}\right)^L \eta^{-P_a} / h(\eta s/t_p) . \tag{34}$$

Substituting the resulting simplified form of $G(\Lambda)$ into (21), the terms involving $P_a$ cancel, leaving

$$\ln(T_q \widehat{T}_q) = -2Lf/KT - 2v_r \tag{35}$$

where $f$, $v_r$ are independent of $L$, being given by

$$-2f/KT = \ln\left(\frac{N(x_q - x_p)(x_q - y_p)(t_p^N - \eta^N)}{(x_q^N - x_p^N)(x_q^N - y_p^N)(t_p - \eta)}\right) + \frac{2}{\pi}\int_S \left(\frac{1}{1-x} - \frac{Nx^{N-1}}{1-x^N}\right)\psi(\Lambda_q, xt_p)\,\mathrm{d}x \tag{36}$$

$$v_r = r\ln\mu_q - \ln A(\omega^{-1/2}mt_q) + \tfrac{1}{2}\ln h(\eta m) + m\int_\eta^{1/\eta} \frac{h'(my)}{\pi h(my)}\,\psi(\Lambda_q, y)\,\mathrm{d}y \tag{37}$$

where

$$m = s/t_p . \tag{38}$$

Note that the RHS of (37) involves $s$ and $p$ only via the ratio $m$.

The expressions (36) and (37) are explicitly real for the Hermitian sub-case mentioned above, $s$ and $m$ then being real and positive. So will be the following alternative formulations. The square roots in (30) and (32), and in the following equation (42), should be chosen to be positive for this case, and continuous.

### 3.3. Free energy

The quantity $f$ is the free energy per site of the alternating superintegrable model in the ferromagnetic phase. Remember that the integral over $S$ arose from an integration around the branch cut of the square root in $g(\Lambda_q, \omega^{1/2}zt_p)$. Going back to this form, provided $|\mathrm{Re}(u_p)| < \pi/(2N)$, we can expand the contour of integration to become the union of the straight-line segments $\arg(z) = \pm\pi/N$. If we define

$$\phi_j(x) = \frac{\sin(\pi j/N)}{\pi[1 + x^2 + 2x\cos(\pi j/N)]} \tag{39}$$

then (36) becomes

$$-2f/KT = \ln\left(\frac{N(x_q - x_p)(x_q - y_p)(t_p^N - \eta^N)}{(x_q^N - x_p^N)(x_q^N - y_p^N)(t_p - \eta)}\right) + 2\int_0^\infty \phi_{N-1}(x)\, g(\Lambda_q, xt_p)\,\mathrm{d}x . \tag{40}$$

For the homogeneous superintegrable case, $x_p = y_p = \eta^{1/2}$, $\mu_p = 1$ and (40) agrees with [6, equation (23)] and [10, equation (6.13)], $k', x, y, G, \rho, \tan(\theta/2), \mathrm{d}\phi$ therein being in our notation $k', x_q/x_p, y_p/y_q, (1-k')(\Lambda_q+1)/(\Lambda_q-1), (x_q - x_p)/(x_q^N - x_p^N)$, $x^{-N/2}, -\pi\phi_{N-1}(x)\,\mathrm{d}x$. In fact, for the more general alternating case (with $p' \neq p$), equation (40) should be contained in the result [10, equation (7.2)] for the row-inhomogeneous superintegrable model, but we have not verified this explicitly.

## 4. Interfacial tension

Just as (40) can be obtained from (36) by expanding the contour of integration to $\arg(z) = \pm\pi/N$, so can we change (37) to

$$v_r = r\ln\mu_q - \ln A(\omega^{-1/2}mt_q) + \tfrac{1}{2}\ln h(\eta m) + m\int_0^\infty \left[\phi_{r-1}(my) + \phi_{r+1}(my)\right] g(\Lambda_q, y)\,\mathrm{d}y\,. \tag{41}$$

This transformation is valid for $r = 0, \ldots, N-2$; for $r = N-1$ the function $h(x)$ has zeros at $x = \omega^{\pm 1/2}$, so the expanded contour has to be indented to avoid the resulting poles. The net effect is that (41) remains valid for $r = N-1$, provided we re-define $\phi_N(x)$ to be $\delta(x-1)$. This is consistent with the $j \to N$ limit of (39).

Yet another formulation of $v_r$ can be obtained by considering the integral of

$$\frac{m\omega^{-1/2}A'(\omega^{-1/2}my)}{\pi\, A(\omega^{-1/2}my)}\,\psi(\Lambda_q, y)$$

over the contour in the $y$-plane shown in figure 3. (There is a logarithmic branch cut from the origin to $t_q$ coming from the $\psi$ function.) The contributions from the segments $AB$, $CD$, $EF$, $GH$, $HI$ give the RHS of (37). Since the integrand is analytic inside the contour, the RHS of (37) is therefore the negative of the contributions from $BC$, $DE$, $FG$, $IA$, so

$$v_r = 2m\int_0^\eta \phi_r(my)\tanh^{-1}\left[\frac{\Lambda_q+1}{\Lambda_q-1}\left(\frac{\eta^N - y^N}{\eta^{-N} - y^N}\right)^{1/2}\right]\mathrm{d}y$$
$$+\ 2m\int_{1/\eta}^\infty \phi_r(my)\tanh^{-1}\left[\frac{\Lambda_q-1}{\Lambda_q+1}\left(\frac{y^N - \eta^{-N}}{y^N - \eta^N}\right)^{1/2}\right]\mathrm{d}y\,. \tag{42}$$

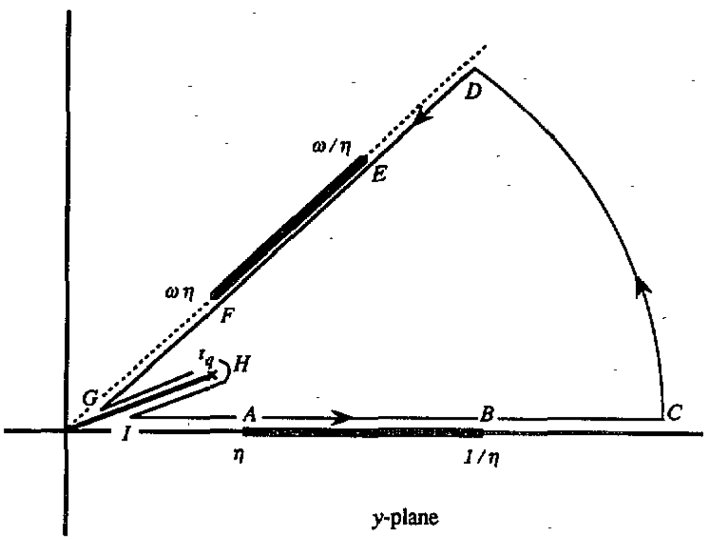


Figure 3. The contour *ABCDEFGHI* in the $y$-plane.

The results (36)–(42) are analytic provided $|\Lambda_q| > 1$ and the arguments of $m, t_p, \omega^{-1/2}t_q$ all lie between $-\pi/N$ and $\pi/N$. For the Ising case, when $N = 2$ and $r = 1$, the RHS of (42) can be evaluated, giving

$$\exp(-v_1) = \frac{2k'^{1/2}k^{-1}(1 - \mathrm{i}mt_q)}{[(1 - t_q^2\eta^{-2})(m^2 + \eta^2)]^{1/2} + [(1 - t_q^2\eta^2)(m^2 + \eta^{-2})]^{1/2}} \,. \qquad (43)$$

We have considered only a particular set of eigenvalues of $\mathbf{T}_q\widehat{\mathbf{T}}_q$, namely those satisfying (25) and (26), $s$ taking all positive values allowed by (28). At sufficiently low temperatures (i.e. for $k'$ small) we expect from [19] that for $M$ large this set will dominate the sum in (6). The free energy is independent of $r$, so from (35)

$$Z_r/Z_0 = \sum \mathrm{e}^{-2Mv_r} \qquad (44)$$

the sum being over all allowed values of $s$.

When $L$ becomes large, the allowed values of $s$ form a dense distribution along the positive real axis and the sum in (6) becomes

$$Z_r/Z_0 = L\int_0^\infty R(s)\,\mathrm{e}^{-2Mv_r}\,\mathrm{d}s \qquad (45)$$

where $v_r$ is defined as a function of $s$ by (37), (41) or (42). All we need to know about the distribution function $R(s)$ is that it is positive and independent of $L$ and $M$.

For the Hermitian sub-case, $v_r$ is a real function of $s$ and has a minimum in the interval $0 < s < \infty$. (In particular, at the symmetric point $t_q = \omega^{1/2}$, $(\Lambda_q - 1)/(\Lambda_q + 1) = \eta^{N/2}$, $t_p = 1$, $v_r$ is symmetric under $s \to 1/s$ and the minimum occurs when $s = 1$.) Denote the minimum (turning) value as $(v_r)_{\mathrm{turn}}$. Taking the limit of $L$ and $M$ large, the integral in (45) is dominated by the contribution from the neighbourhood of the minimum, so we see from (7) that

$$\epsilon_r/K\mathcal{T} = 2(v_r)_{\mathrm{turn}} \,. \qquad (46)$$

In general, $v_r$ is not real and we have to proceed as in [26] and [27]. The integral in (45) can be evaluated for large $M$ by the method of steepest descents, first deforming the contour of integration in the complex $s$-plane so as to pass through the appropriate saddle point of the function $v_r$ (i.e. the point in the $s$-plane where its complex derivative vanishes). *Thus the interfacial tension is still given by (46), provided we take $(v_r)_{\mathrm{turn}}$ to be the value of $v_r$ at this saddle point.*

### *4.1. Continuation back to the real case*

We have obtained these results (37), (41), (42) and (46) for the alternating superintegrable case, with $u_p$ in the vicinity of the branch point $-\frac{1}{2}\mathrm{i}\ln\eta$ and $u_q$ on or near the real axis, with $0 < \mathrm{Re}(u_q) < \pi/N$. Now recall that $v_r$ depends on $s$ and $p$ only via the ratio $m = s/t_p$. Thus changing $p$ only re-scales the argument $m$ and does not affect the saddle-point value of $v_r$. Hence for the alternating superintegrable model *the interfacial tension is independent of $p$*. This is in agreement with the $Z$-invariance argument mentioned in section 2. In fact, using that argument and analytically continuing back to the real case, we expect (37), (41), (42) and (46) to be true for the general chiral Potts model when $u_p, u_{p'}, u_q$ satisfy (13) and

$0 < u_q < \pi/N$. We also expect them to be true for the Hermitian case, at least provided $u_q$ is not too far from the real axis.

More explicitly, $v_r$ is defined as a function $v_r(m)$ of $m$ by (37), (41) and (42). Let $m_0$ be the value of $m$ at which $v_r{}'(m)$ vanishes, such that $m_0$ is a continuous function of $q$, positive real when $2u_q - \pi/N$ is pure imaginary, and unimodular when $u_q$ is real. (For the symmetric point $u_q = \pi/(2N)$ we have $m_0 = 1$.) Then the interfacial tension $\epsilon_r$ is given by

$$\epsilon_r/KT = 2\,v_r(m_0)\,. \tag{47}$$

It depends on the horizontal rapidity $q$, but not on the alternating vertical rapidities $p$ and $p'$. It can be analytically continued to all real values of $u_q$. Rotating the lattice through 180° is equivalent to incrementing $u_q$ by $\pi/N$ and replacing $v_q$ by $(\pi/N) - v_q$. (This is the automorphism $q \to Rq$ defined in [1].) The effect on (42) and (47) is to negate $m_0$ and to replace $\epsilon_r$ by $\epsilon_{N-r}$. This is indeed consistent with the 180° rotation symmetry.

In general (guided by the near-critical case discussed below), we expect $\mathrm{i}m_0\,\mathrm{e}^{-\mathrm{i}Nu_q}$ to lie in the RHP, probably close to the positive real axis.

Again, the Ising case $N = 2$ is simple and illuminating. The RHS of (43) is stationary when $m = m_0 = -\mathrm{i}t_q$, with value $k'^{1/2}$, so

$$\epsilon_1/KT = -\ln k' \qquad 0 < k' < 1\,. \tag{48}$$

This agrees with [24, equation (7.10.18)], the $k$ therein being our $k'$.

### 4.2. Critical behaviour

The system becomes critical as $k' \to 1$ and $k$, $\eta \to 0$. The third form, equation (42), of $v_r$ is ideal for investigating this limit. From it and (9), keeping $m$ and $t_q$ fixed, we deduce that for $k$ small

$$v_r(m) = \rho\,\eta^{(N+2)/2}\,\sin(\pi r/N)\,\mathrm{i}(\mathrm{e}^{-\mathrm{i}Nu_q}m - \mathrm{e}^{\mathrm{i}Nu_q}/m) \tag{49}$$

where

$$\rho = (2/\pi)\int_0^1 (1 - x^N)^{1/2}\,\mathrm{d}x = 2\beta\left(\frac{1}{N}, \frac{1}{2}\right)/\pi(N+2)\,. \tag{50}$$

This function is stationary when $m = m_0 = -\mathrm{i}\,\mathrm{e}^{\mathrm{i}Nu_q}$, so from (47)

$$\epsilon_r/KT \doteq 4\rho\,\eta^{(N+2)/2}\sin(\pi r/N)\,. \tag{51}$$

(For $N = 2$ and $r = 1$, then $\rho = \frac{1}{2}$ and the RHS simplifies to $k^2/2$, in agreement with (48).)

This result (51) gives the interfacial tension in the scaling region near criticality. Note that it is independent of the horizontal rapidity $q$, so that it is unchanged by rotating the lattice through 90°. (For the isotropic near-critical $N = 2$ Ising case it is also true that $\epsilon_1 = \sqrt{2}\,\sigma'$, where $\sigma'$ is the interfacial tension obtained by Onsager [28]. From our point of view $\sqrt{2}\,\sigma'$ is the interfacial tension per unit length in the 45° direction, so this is consistent with $\epsilon_1$ being isotropic in the scaling region.)

In [19] we noted that in the low-temperature limit $k' \to 0$ that

$$\epsilon_j < \epsilon_k + \epsilon_l \qquad j = k + l \mod N \tag{52}$$

for $0 < k, l < N$. This means that it is never energetically favourable to interpose an intermediate phase between two others, so there is no wetting. From (51) it is readily verified that this is still true in the scaling region, and therefore probably throughout the ordered regime $0 < k' < 1$.

From scaling theory we expect that

$$\epsilon_r \propto (\mathcal{T}_c - \mathcal{T})^\mu \tag{53}$$

$\mathcal{T}$ being the temperature, $\mathcal{T}_c$ its critical value and $\mu$ the critical exponent. For the chiral Potts model, near criticality the Boltzmann weights are linear in the variable $k^2$, or equivalently $\eta^N$, so we can take $(\mathcal{T}_c - \mathcal{T})$ to be $\eta^N$ and we see from (51) that $\mu$ is given by (1).

Thus $\mu$ is independent of $r$ and is unchanged by rotations through 90°. These results differ from those given in [10]: there, $s'_Q$ is the vertical tension with free boundary top and bottom spins, $Q$ being our $r$. From low-temperature calculations along the lines of those in [19], it is apparent that this change of boundary condition has a significant effect on the vertical interfacial tension, since the interface no longer has to finish in the top row at the same point as it started at the bottom. For instance, it makes $\epsilon_r$ proportional to $r$, so it becomes energetically neutral to interpose an intermediate phase between two others. As is remarked after equation (6.25) therein, $s_Q$ is therefore not the usual interfacial tension. It has critical exponent 1.

The other result reported in [10] is the interfacial tension $s_a$ for the homogeneous superintegrable model with fixed boundary spins at top and bottom. Rotating through 90°, this should be comparable to our result for $\epsilon_r$ when $u_q = -\frac{1}{2}\mathrm{i}\ln\eta$. It is not clear that our results can be extended to this singular point in the $u_q$ plane so far from the real axis. Certainly the critical exponent can be expected to be different, since $t_q$ now tends to zero as we approach criticality. In fact the exponent of $s_a$ is $2/N$, and it is intriguing to note that (1) is simply the arithmetic mean of the two exponents reported in [10]. The specific heat exponent $\alpha$ is $1 - 2/N$ [2]. In [6] we noted that the superintegrable model appears to violate the two-dimensional scaling relation $2\mu = 2 - \alpha$. It is therefore pleasing to note from (1) that for the physical chiral Potts model (i.e. the one with real positive Boltzman weights) we have regained this scaling relation.

Our result also has to be reconciled with the suggestion in [22, 23] that the chiral Potts model is the wetting transition of the Ostlund–Huse model, which presumably implies $\epsilon_{\mathrm{mod}(k+l,N)} = \epsilon_k + \epsilon_l$, in apparent contradiction of our result (52). Those papers consider the interfacial tension in what to us is the 45° direction, whereas we are restricted to the vertical and horizontal directions. It is conceivable that this is the source of the discrepancy, in which case the interfacial tension must be anisotropic for $N > 2$, even in the scaling region.

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